hep-th/yymmxxx

Multigravity from a discrete extra dimension

C. Deffayet[a,c,1], J. Mourad[b,c,2],

[a] *GReCO/IAP[3] 98 bis boulevard Arago, 75014 Paris, France*

[b] *Laboratoire de Physique Théorique[4], Bât. 210 , Université Paris XI, 91405 Orsay Cedex, France*

[c] *Fédération de recherche APC, Université Paris VII, 2 place Jussieu - 75251 Paris Cedex 05, France.*

Multigravity theories are constructed from the discretization of the extra dimension of five dimensional gravity. Using an ADM decomposition, the discretization is performed while maintaining the four dimensional diffeomorphism invariance on each site. We relate the Goldstone bosons used to realize nonlinearly general covariance in discretized gravity to the shift fields of the higher dimensional metric. We investigate the scalar excitations of the resulting theory and show the absence of ghosts and massive modes; this is due to a local symmetry inherited from the reparametrization invariance along the fifth dimension.

[1] deffayet@iap.fr

[2] mourad@th.u-psud.fr

[3] FRE 2435 du CNRS.

[4] UMR 8627 du CNRS.

1 Introduction

Since the work of Pauli and Fierz [1], it is known that the consistency of theories with spin 2 fields is highly non trivial. A manifestly covariant lagrangian formulation relies on a symmetric rank 2 tensor which contains a priori ten degrees of freedom. Using kinetic terms with four local invariances results in constraints which diminish the number of degrees of freedom to six. The additional scalar degree of freedom turns out to be a ghost whose elimination, at the linear level, dictates the Pauli-Fierz quadratic combination. Boulware and Deser [2] have argued that the incorporation of self interaction generically results in the reappearance of the pathologic scalar degree of freedom. The same reasons render the consistency of multigravity theories [3, 4, 5, 6], with many interacting metrics, highly non trivial. Higher dimensional Einstein gravity when compactified to four dimensions results in a finite number of massless modes among whom is a graviton and an infinite tower of massive modes. The truncation to a finite number of modes is however inconsistent [7, 8]. On the other hand a procedure has been recently proposed to obtain from five dimensional Yang-Mills theories a four dimensional theory which approach the five dimensional one in the infrared and which has a finite number of modes. The key point in the approach is to replace the extra component of the vector field by bifundamental scalars which can be viewed as arising from the latticized Wilson line along the fifth dimension. The goal of the present letter is to present such a construction for five dimensional gravity and to see to which extend the consistency of the higher dimensional theory descends to the discretized version.

Our starting point will be the 4+1 ADM decomposition [9] of the five dimensional metric, which will be much more convenient than the Kaluza-Klein splitting. In particular we will show the analogy between the shift vector in the gravity side and the fifth component of the gauge field in the Yang-Mills side. The bifundamental scalars in the Yang-Mills side can be seen as providing a mapping from fundamentals of a gauge group on a site to the fundamentals of the gauge group on the neighbouring site. We will show that the corresponding object in the gravity side is a mapping from one four dimensional manifold located on a site in the discrete extra dimension to the neighbouring one. The map reduces in the continuum limit to the shift vector of the ADM decomposition.

The paper is organized as follows. The second Section is devoted to the introduction of the (4+1) ADM decomposition of the 5D metric and the transformations of the various resulting fields under the 5D diffeomorphisms. In the third Section we show how to discretize the 5D Einstein-Hilbert action while maintaining 4D diffeomorphism invariance on each site. We relate the link fields [10] used to realize non linearly 4D general covariance to the shift vector field of the ADM formalism. In the fourth Section we discuss the spectrum of the resulting action. We exhibit, at the quadratic level, an additional local symmetry inherited from the reparametrization invariance along the fifth dimension. This symmetry is crucial to show the absence of ghosts and massive scalar modes. Our analysis suggests to reconsider previous work [11] on strong coupling effects in these theories.

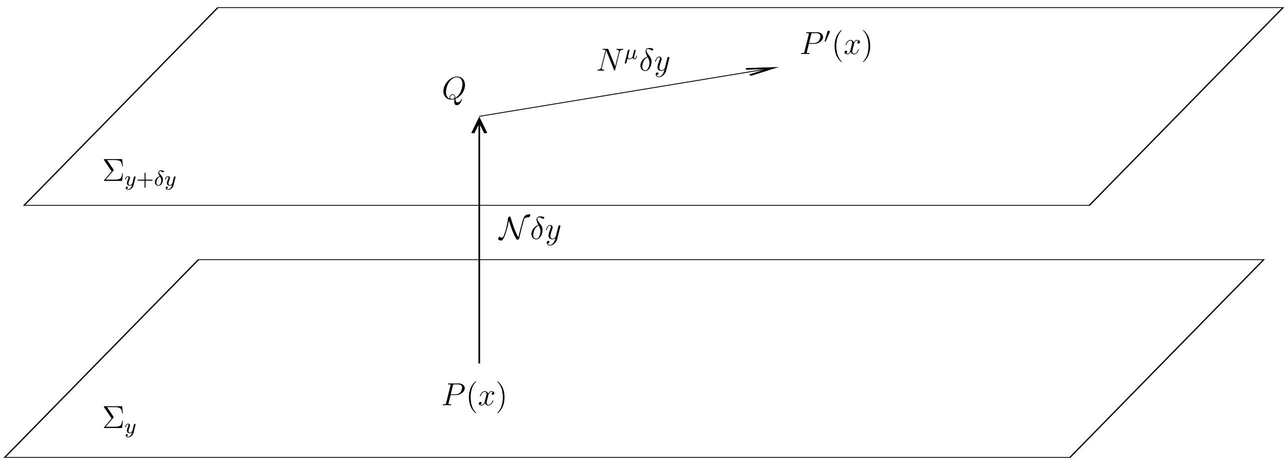

Figure 1: The points P and P' have 4D coordinates x^μ, Q is on the normal to Σ_y at P.

2 ADM decomposition and invariances

Let us briefly recall the ADM splitting of the five dimensional metric $\tilde{g}$ and the corresponding expression for the Einstein-Hilbert action

$$S_{EH} = M_{(5)}^3 \int d^5X \sqrt{-\tilde{g}}\tilde{R}. \tag{1}$$

Above, and in the following, we use expressions with a *tilde* for denoting quantities of the 5D continuum theory, like the 5D metric $\tilde{g}_{AB}$, upper case Latin letters from the beginning of the alphabet, A, B, C... to denote 5D indices and lower case Greek letters, $\mu, \nu, \alpha, ...$ will be denoting 4D indices. We consider a foliation of the five dimensional manifold by four dimensional ones, Σ_y, located at given y. Each slice has its four dimensional metric $g_{\mu\nu}(x, y)$. The distance between the manifolds Σ_y and $\Sigma_{y+\delta y}$ is denoted by $\mathcal{N}\delta y$ defining the lapse field $\mathcal{N}$. The normal to Σ_y at a point with 4D coordinates x^μ hits $\Sigma_{y+\delta y}$ at a point with coordinates which in general differ from x^μ by a vector $N^\mu \delta y$ which defines the shift field (see figure 1). In brief, the fields $\mathcal{N}$, N_μ and $g_{\mu\nu}$ are related to components of the 5D metric $\tilde{g}_{AB}$ by

$$\tilde{g}_{\mu\nu} = g_{\mu\nu}, \tag{2}$$
$$\tilde{g}_{\mu y} = N_\mu \equiv g_{\mu\alpha}N^\alpha, \tag{3}$$
$$\tilde{g}_{yy} = \mathcal{N}^2 + g_{\mu\nu}N^\mu N^\nu. \tag{4}$$

After an integration by part, the 5D Einstein-Hilbert action (1) can be written as

$$M_{(5)}^3 \int d^4x dy \sqrt{-g}\mathcal{N}\left\{R + K_{\mu\nu}K_{\alpha\beta}\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\right\}, \tag{5}$$

where $K_{\mu\nu}$ is the extrinsic curvature of surfaces Σ_y:

$$K_{\mu\nu} = \frac{1}{2\mathcal{N}}\left(g'_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu\right), \tag{6}$$

where D_μ is the covariant derivative associated with the induced metric $g_{\mu\nu}$ and a prime denotes an ordinary derivative with respect to y. The equations of motions derived from action (5) varying with respect to $\mathcal{N}$, N^μ and $g^{\mu\nu}$ yield respectively

$$
\begin{aligned}
R &= K^2 - K^\rho_\sigma K^\sigma_\rho && (7)\\
D^\mu K &= D^\nu K^\mu_\nu && (8)\\
G_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}\left(K^2 - K^\rho_\sigma K^\sigma_\rho\right) + \frac{D_\mu D_\nu \mathcal{N} - g_{\mu\nu} D^\rho D_\rho \mathcal{N}}{\mathcal{N}} - g_{\rho\mu} g_{\sigma\nu} \frac{\partial_y \left\{\sqrt{-g}\left(K g^{\rho\sigma} - K^{\rho\sigma}\right)\right\}}{\mathcal{N}\sqrt{-g}} \\
&\quad -\frac{2}{N}\left\{D_\nu\left(N_\mu K\right) - D_\rho\left(K^\rho_\nu N_\mu\right) - \frac{1}{2}g_{\mu\nu} D^\rho\left(K N_\rho\right) + \frac{1}{2}D^\rho\left(N_\rho K_{\mu\nu}\right)\right\} \\
&\quad +2\left(K^\rho_\mu K_{\rho\nu} - K K_{\mu\nu}\right), && (9)
\end{aligned}
$$

where $G_{\mu\nu}$ is the Einstein tensor for four dimensional metric $g_{\mu\nu}$, and K is defined by $K \equiv K_{\mu\nu} g^{\mu\nu}$.

We now turn to determine the transformation properties of the 4D fields under the various diffeomorphisms. These are generated by the vector fields

$$\tilde{\xi} = \xi^A \partial_A = \xi^\mu \partial_\mu + \xi^5 \partial_y = \xi + \xi^5 \partial_y, \tag{10}$$

and act on the metric as

$$\delta \tilde{g}_{AB} = \mathcal{L}_{\tilde{\xi}} \tilde{g}_{AB} = \tilde{\xi}^C \partial_C \tilde{g}_{AB} + \tilde{g}_{AC} \partial_B \tilde{\xi}^C + \tilde{g}_{CB} \partial_A \tilde{\xi}^C, \tag{11}$$

where $\mathcal{L}_{\tilde{\xi}}$ is the Lie derivative with respect to $\tilde{\xi}$. It will be very useful to write the corresponding transformations for the 4D metric, the lapse and the shift fields. If we define the shift vector by

$$\bar{N} = N^\mu \partial_\mu, \tag{12}$$

and the *covariant* derivative, D_y, by

$$D_y = \partial_y - \bar{N}, \tag{13}$$

then, under y dependent four dimensional diffeomorphisms, that is for $\xi^5 = 0$, we have

$$
\begin{aligned}
\delta g_{\mu\nu} &= \mathcal{L}_\xi g_{\mu\nu}, && (14)\\
\delta \bar{N} &= [D_y, \xi] = \partial_y \xi - [\bar{N}, \xi], && (15)\\
\delta \mathcal{N} &= \xi(\mathcal{N}). && (16)
\end{aligned}
$$

The transformation of $g_{\mu\nu}$ and $\mathcal{N}$ is as expected the one of a 4D metric and scalar respectively. The transformation of $\bar{N}$ has however an additional term with respect to the usual one characterizing the transformation of a vector. This new term is reminiscent of the inhomogeneous term contributing to the transformation of a gauge field. Indeed this analogy justifies the covariant derivative name we gave to D_y: suppose for example that ϕ is a 5D scalar and consider $\partial_y \phi$, it is not a scalar under a diffeomorphism generated by ξ

$$\delta \partial_y \phi = \partial_y \xi(\phi) = \xi(\partial_y \phi) + (\partial_y \xi)(\phi), \tag{17}$$

whereas $D_y\phi$ is indeed a scalar

$$\delta D_y\phi = \delta\partial_y\phi - (\delta\bar{N})(\phi) - \bar{N}(\delta\phi) = \xi(D_y\phi), \tag{18}$$

where we used the transformation rule of $\bar{N}$ (15). Similarly if T is a tensor then $\mathcal{L}_{D_y}T$ is also a tensor under 4D y-dependent diffeomorphisms. One can thus view the role of $\bar{N}$ as rendering possible the formulation of an action invariant under y dependent 4D diffeomorphisms. Incidentally, this remark tells us how the shift fields enter into the equations of motion: it suffices to replace everywhere ∂_y by D_y. Notice also that, using D_y, the extrinsic curvature can be simply expressed as $K_{\mu\nu} = (\mathcal{L}_{D_y}g_{\mu\nu})/(2\mathcal{N})$.

We next consider diffeomorphisms along the fifth dimension. In fact it is more convenient to consider diffeomorphisms generated by D_y, that is $\tilde{\xi} = \zeta D_y$, with ζ depending on y as well as on x. A short calculation gives the following rules

$$\begin{aligned} \delta g_{\mu\nu} &= \zeta\mathcal{L}_{D_y}g_{\mu\nu}, \quad \delta\mathcal{N} = D_y(\zeta\mathcal{N}), && (19)\\ \delta N^\mu &= \mathcal{N}^2 g^{\mu\nu}\partial_\nu\zeta. && (20) \end{aligned}$$

To end this section, let us also mention that it is also convenient to write the action (5) in the "Einstein frame" (from the point of view of the 4D metric $g_{\mu\nu}$). This can be achieved by performing the Weyl rescaling $g_{\mu\nu} = \exp\left(-\frac{\phi}{\sqrt{3}}\right)\gamma_{\mu\nu}$, with $\mathcal{N} = \exp\left(\frac{\phi}{\sqrt{3}}\right)$. Under this transformation the action (5) is rephrased into

$$M_{(5)}^3\int d^4x dy\sqrt{-\gamma}\left\{R(\gamma) - \frac{1}{2}\gamma^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + e^{-\sqrt{3}\phi}Q_{\mu\nu}Q_{\alpha\beta}\left(\gamma^{\mu\nu}\gamma^{\alpha\beta} - \gamma^{\mu\alpha}\gamma^{\nu\beta}\right)\right\}, \tag{21}$$

with

$$\begin{aligned} Q_{\mu\nu} &= \frac{1}{2}\left\{\gamma'_{\mu\nu} - \gamma_{\mu\nu}\frac{\phi'}{\sqrt{3}} - \nabla_\mu V_\nu - \nabla_\nu V_\mu + \gamma_{\mu\nu}V^\rho\nabla_\rho\frac{\phi}{\sqrt{3}}\right\} && (22)\\ &= \frac{1}{2}\left[\mathcal{L}_{D_y}\gamma_{\mu\nu} - \frac{\gamma_{\mu\nu}}{\sqrt{3}}D_y\phi\right], && (23) \end{aligned}$$

V^μ defined by $V^\mu \equiv N^\mu$, and $V_\mu \equiv \gamma_{\mu\nu}V^\nu$.

3 Discretization

We first review the deconstruction of gauge theories [12]. Consider a 5D non-abelian gauge field $A = A_A^a t_a dx^A \equiv A_\mu dx^\mu + A_5 dy$ with gauge group, e.g., $SO(M)$. Under a y dependent gauge transformation the transformation rules are

$$A' = uAu^{-1} - udu^{-1}, \tag{24}$$

where u is an element of $SO(M)$. These reduce to the 4D y-dependent transformations for $A_\mu dx^\mu$ and A_5, which is a scalar viewed from 4D, has the following transformation

$$A_5' = uA_5u^{-1} - u\partial_y u^{-1}, \tag{25}$$

which gives for an infinitesimal transformation

$$\delta A_5 = \partial_y \epsilon - [A_5, \epsilon]. \tag{26}$$

Note the formal analogy between (26) and (15): the Lie bracket in (15) is replaced in (26) by a matrix commutator.

It is very convenient when discretizing the y dimension to replace A_5 by the Wilson line [13, 14]

$$W(y', y) = P e^{\int_y^{y'} A_5 dy}, \tag{27}$$

which transforms as

$$W'(y', y) = u(y') W(y', y) u^{-1}(y). \tag{28}$$

The lattice version of the gauge theory is now easily obtained: one has N sites, on each site a gauge field with a corresponding gauge group $SO(M)_i$ and on each link between neighbouring sites a scalar $W(y_i, y_{i+1})$ transforming in the bifundamental of the gauge groups $SO(M)_i \times SO(M)_{i+1}$. One can then write an effective action for the gauge fields and the scalars. The continuum limit is recovered when the number of sites goes to infinity and a goes to zero; the vacuum expectation value of the scalars is then the identity:

$$W(i, i+1) = 1 - a A_5(y) + \ldots, \tag{29}$$

where a is the lattice spacing. In the broken phase one has a massless gauge boson corresponding to the diagonal subgroup and a collection of massive spin one particles with masses

$$m_k = a^{-1} \sin \frac{k\pi}{N}, k = 1, \ldots N-1. \tag{30}$$

These reproduce, in the infrared, the Kaluza-Klein spectrum of the first modes when the radius is given by aN.

We noted previously that $\bar{N}$ has a transformation law which is similar to the fifth component of the gauge potential, the gauge group being replaced by the diffeomorphism group. While the gauge field can be represented by a matrix this is no longer true for $\bar{N}$ which is rather to be thought of as an operator acting on functions on a four dimensional manifold. The gauge Wilson line (27) and the analogy between $\bar{N}$ and A_5 motivates the consideration of

$$W(y', y) = P \exp \int_y^{y'} dz \bar{N}, \tag{31}$$

or more explicitly

$$\begin{aligned} W_{y',y} &= 1 + \int_y^{y'} dz N^\mu(z) \partial_\mu + \int_y^{y'} dz_1 N^{\mu_1}(z_1) \partial_{\mu_1} \int_y^{z_1} dz_2 N^{\mu_2}(z_2) \partial_{\mu_2} + \ldots \\ &\quad + \int_y^{y'} dz_1 N^{\mu_1}(z_1) \partial_{\mu_1} \int_y^{z_1} dz_2 N^{\mu_2}(z_2) \partial_{\mu_2} \ldots \int_y^{z_{p-1}} dz_p N^{\mu_p}(z_p) \partial_{\mu_p} + \ldots \end{aligned} \tag{32}$$

Now $W(y', y)$ defines a mapping from functions (scalar fields) on Σ_y to functions (scalar fields) on $\Sigma_{y'}$. Explicitly, let $\phi(x)$ be a scalar field defined on the hypersurface Σ_{y_0} and consider

$$\phi_y = W(y, y_0)(\phi). \tag{33}$$

Then ϕ_y verifies the equation $\partial_y \phi_y = \bar{N}_y(\phi_y)$ and is subject to the boundary condition $\phi_{y_0} = \phi$. Let $\xi(y)$ generate a y-dependent 4D diffeomorphism then from the transformation of $\bar{N}$ given in (15) we get

$$\delta W(y', y) = \xi(y')W(y', y) - W(y', y)\xi(y), \tag{34}$$

which implies that indeed ϕ_y defined in (33) transforms under diffeomorphisms as $\delta\phi_y = \xi(y)(\phi_y)$ if ϕ transforms as $\delta\phi = \xi(y_0)(\phi)$. A convenient and useful way of writing (33) is

$$\phi_y = \phi \circ X(y, y_0), \tag{35}$$

where $X(y, y_0)$ is a mapping from the manifold Σ_y to Σ_{y_0} generated by $\bar{N}$, that is

$$\partial_y X^\mu(y, y_0; x) = N_y^\mu(x), \quad X^\mu(y_0, y_0; x) = x^\mu, \tag{36}$$

which can be written as

$$\begin{aligned} X^\mu(y, y_0; x) &= W(y, y_0)(x^\mu), &(37)\\ &= x^\mu + \int_{y_0}^{y} dz N^\mu(z; x) + \int_{y_0}^{y} dz_1 N^\nu(z_1; x) \int_{y_0}^{z_1} \partial_\nu(N^\mu(z_2; x)) + \ldots &(38) \end{aligned}$$

the right hand side of the first line being understood as the action of the W on the function x^μ.

It is possible to extend W so that it maps tensors of arbitrary rank on Σ_y to tensors on $\Sigma_{y'}$. This is done with the help of the Lie derivative as follows

$$W(y', y) = P \exp \int_{y}^{y'} dz L_{\bar{N}}. \tag{39}$$

It reduces to the previous expression (31) when acting on scalars. The Leibniz rule for the Lie derivative results in a simple action of W on the direct product of tensor:

$$W(y', y)(T_1 \otimes T_2) = [W(y', y)(T_1)] \otimes [W(y', y)(T_2)], \tag{40}$$

where T_1 and T_2 are arbitrary tensors on Σ_y. The commutation of the Lie derivative and the exterior derivative when acting on forms translates also to the simple property

$$d[W(y', y)(\omega_y)] = W(y', y)(d\omega_y), \tag{41}$$

where ω_y is an arbitrary form defined on Σ_y. The geometric interpretation of the map W is clear from figure 1: when y and y' are infinitesimally close W maps the point P with coordinates x on Σ_y to the point Q with coordinates $x^\mu + \delta y N^\mu$ on $\Sigma_{y+\delta y}$.

We are now in a position of performing the discretization of the Einstein-Hilbert action along the y direction. We replace y by ia with i an integer and a the lattice spacing. The fields

are thus the metric on each site $g^{(i)}_{\mu\nu}$, the lapse fields $\mathcal{N}^{(i)}$ and the Wilson line $W(i,i+1)$ which, as in the gauge theory, replaces the shift vector. The y derivative, as we noted in the previous section, appears in the continuum in the combination D_y. The Lie derivative of a tensor field with respect to D_y can be written as

$$\mathcal{L}_{D_y} T_y = \lim_{\delta y \to 0} \frac{W(y, y+\delta y) T_{y+\delta y} - T_y}{\delta y}. \tag{42}$$

¿From this we see that the simplest discrete counterpart of the Lie derivative along D_y is

$$\Delta_{\mathcal{L}} T_i = \frac{W(i,i+1) T_{i+1} - T_i}{a}. \tag{43}$$

It is now immediate to get the discretized Einstein-Hilbert action from (5)

$$S = M^3_{(5)} a \sum_i \int d^4x \sqrt{-g_i} \mathcal{N}_i \left[R(g_i) + \frac{1}{4\mathcal{N}_i^2} (\Delta_{\mathcal{L}} g_i)_{\mu\nu} (\Delta_{\mathcal{L}} g_i)_{\alpha\beta} \left(g_i^{\mu\nu} g_i^{\alpha\beta} - g_i^{\mu\alpha} g_i^{\nu\beta} \right) \right] \tag{44}$$

The action is invariant under the product of all diffeomorphism groups associated to the points of the lattice. Under such a transformation generated by ξ_i, the different fields transform as

$$\delta g_i = L_{\xi_i} g_i, \quad \delta \mathcal{N}_i = \xi_i(\mathcal{N}_i), \quad \delta W(i,i+1) = \xi_i W(i,i+1) - W(i,i+1)\xi_{i+1}. \tag{45}$$

These reduce in the continuum limit to (14), (16) and (34). The explicit expression of the components of $W(i,i+1)T_{i+1}$ can be easily written down with the help of $X^\mu(i,i+1;x)$, a mapping between the manifolds at i and $i+1$ which is the discrete counterpart of $X^\mu(y, y_0; x)$ defined in (38). In fact, we have

$$[W(i,i+1)T_{i+1}]_{\mu_1,\dots\mu_r}(x) = T_{i+1}(X^\mu(i,i+1;x))_{\nu_1,\dots\nu_r} \partial_{\mu_1} X^{\nu_1} \dots \partial_{\mu_r} X^{\nu_r}. \tag{46}$$

The variation of the action with respect to $W(i,i+1)$ amounts to a variation with respect to the mappings $X^\mu(i,i+1;x)$.

Notice that the action is not the most general action with the symmetries (45) since it descends from a 5D theory which had also a reparametrization invariance along the y direction. This will have very important consequences as we will show in the next section.

The same discretization procedure can be applied to action (21) yielding directly the action with canonical kinetic terms for the metric on each sites

$$\sum_i M^3_{(5)} a \int d^4x \sqrt{-\gamma_{(i)}} \left\{ R(\gamma_{(i)}) - \frac{1}{2} \gamma^{\mu\nu}_{(i)} \nabla_\mu \phi_{(i)} \nabla_\nu \phi_{(i)} + e^{-\sqrt{3}\phi_{(i)}} Q^{(i)}_{\mu\nu} Q^{(i)}_{\alpha\beta} \left(\gamma^{\mu\nu}_{(i)} \gamma^{\alpha\beta}_{(i)} - \gamma^{\mu\alpha}_{(i)} \gamma^{\nu\beta}_{(i)} \right) \right\}, \tag{47}$$

with

$$Q^{(i)}_{\mu\nu} = \frac{1}{2} \left\{ \Delta_{\mathcal{L}} \gamma^{(i)}_{\mu\nu} - \gamma^{(i)}_{\mu\nu} \frac{\Delta_{\mathcal{L}} \phi^{(i)}}{\sqrt{3}} \right\}. \tag{48}$$

Before closing this section let us note that a way to realize diffeomorphism invariance on every site has been proposed in ref. [10] with no a priori relation to an extra dimension. The

procedure amounts to the incorporation of maps between interacting sites. This is the analog of our $W(i,i+1)$. Our construction shows that this map arises naturally in the discretization procedure and allows to identify the continuum limit of the map with the shift vector of the ADM decomposition. In fact in the continuum limit we have, from (38)

$$X^\mu(i,i+1,x) = x^\mu + aN^\mu(y;x) + O(a^2). \tag{49}$$

Different approaches to the deconstruction of gravity theories relying mainly on the local Lorentz invariance in the Cartan moving basis formalism have been considered in [15].

4 Spectrum of the action

This section is devoted to the determination of the propagating modes that are contained in the action (44) at the quadratic level. Let us first make a naive counting of the degrees of freedom (d.o.f.) that arise from a generic action with the symmetries we explicitly implemented in the action 44, with a finite number, N, of sites. We started with $N\times 10$ d.o.f. in the $4D$ metrics, N lapse fields and $(N-1)\times 4$ d.o.f. in the mappings $W(i,i+1)$. The total number of d.o.f. is thus $15\times N-4$. The action has local invariances with $4N$ parameters due to the 4D diffeomorphism on the N manifolds, this reduces the number of d.o.f. to $(15\times N-4)-(4+4)N = 7N-4$. Out of these we expect to have one graviton (2 d.o.f.) and $N-1$ massive spin 2 particles (5N-5 d.o.f.). The remaining degrees of freedom are expected to be shared by a number of zero modes (scalars and vectors), which does not depend on N but depends on the boundary conditions, as well as a number of massive scalars. The latter number depends on N as $2N+c$, where c is a constant which depends on the boundary conditions but which does not depend on N. These scalars are potentially pathologic, they may lead to ghosts or tachyons. For a generic multigravity theory, ghosts and instabilities do indeed appear [3, 2]. The higher dimensional theory we started with, before discretization, does not have these pathologies. It is thus possible that the action (44) inherited the consistency of the continuum action. The rest of this section is devoted to the proof that this is indeed the case at least to the quadratic order in the fluctuations. We will find that the massive scalar modes decouple at the quadratic level. This is due to an extra local symmetry which removes $2N-2$ degrees of freedom.

In order to get the standard kinetic terms for the metrics, let us first perform a Weyl rescaling on the metric in (44). So we define the metrics $\gamma^{(i)}_{\mu\nu}$ and the scalars $\phi^{(i)}$ by $g^{(i)}_{\mu\nu} = \exp\left(-\frac{\phi_{(i)}}{\sqrt{3}}\right)\gamma^{(i)}_{\mu\nu}$, $\mathcal{N}^{(i)} \equiv \exp\left(\phi^{(i)}/\sqrt{3}\right)$. The next step is to expand the action around the vacuum

$$\gamma^{(i)}_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_p}h^{(i)}_{\mu\nu}, \quad \phi^{(i)} = \frac{\varphi^{(i)}}{M_p}, \quad X^\mu(i,i+1) = x^\mu + \frac{a}{M_p}n^\mu_{(i)}, \tag{50}$$

and to keep the quadratic fluctuations in the fields. In (50), M_p is given by $M_p^2 = M_{(5)}^3 a$. We obtain

$$\int d^4x \sum_i \frac{1}{4}\left\{\partial_\rho h^{\mu\nu}_{(i)}\partial_\sigma h^{\alpha\beta}_{(i)}\left(\eta^{\rho\sigma}\eta_{\mu\nu}\eta_{\alpha\beta} - \eta^{\rho\sigma}\eta_{\mu\alpha}\eta_{\nu\beta} + 2\delta^\sigma_{(\nu}\eta_{\mu)\beta}\delta^\rho_\alpha - \eta_{\mu\nu}\delta^\sigma_\beta\delta^\rho_\alpha - \eta_{\alpha\beta}\delta^\sigma_\nu\delta^\rho_\mu\right)\right.$$

$$
+\left(\Delta\left(h^{(i)}_{\mu\nu}-\frac{\eta_{\mu\nu}}{\sqrt{3}}\varphi^{(i)}\right)-2\partial_{(\mu}n^{(i)}_{\nu)}\right)\left(\Delta\left(h^{(i)}_{\alpha\beta}-\frac{\eta_{\alpha\beta}}{\sqrt{3}}\varphi^{(i)}\right)-2\partial_{(\alpha}n^{(i)}_{\beta)}\right)\left(\eta^{\mu\nu}\eta^{\alpha\beta}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)
$$

$$
-2\partial_\mu\varphi^{(i)}\partial_\nu\varphi^{(i)}\eta^{\mu\nu}\Big\}. \tag{51}
$$

where we have defined the finite difference operator Δ acting on a field $\mathcal{F}_{(i)}$ as

$$
\Delta\mathcal{F}_{(i)}=\frac{\mathcal{F}_{(i+1)}-\mathcal{F}_{(i)}}{a}, \tag{52}
$$

two spacetime indices between a parenthesis indicates a symmetrization on these indices weighted by a factor two [5], we have $h^{\alpha\beta}\equiv\eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}$ and $h\equiv h^{\alpha\beta}\eta_{\alpha\beta}$. The equations of motion derived from this action read

$$
\begin{aligned}
0 &= \partial^\alpha\partial_\mu h^{(i)}_{\alpha\nu}+\partial^\alpha\partial_\nu h^{(i)}_{\alpha\mu}+\eta_{\mu\nu}\Box h^{(i)}-\Box h^{(i)}_{\mu\nu}-\eta_{\mu\nu}\partial^\alpha\partial^\beta h^{(i)}_{\alpha\beta}-\partial_\mu\partial_\nu h^{(i)} \\
&\quad +\Delta\left(\Delta\left(h_{\alpha\beta}-\frac{\eta_{\alpha\beta}}{\sqrt{3}}\varphi\right)-2\partial_{(\alpha}n_{\beta)}\right)_{(i-1)}\left(\eta_{\mu\nu}\eta^{\alpha\beta}-\delta^\alpha_\mu\delta^\beta_\nu\right) && (53)\\
0 &= \partial_\mu\left(\Delta\left(h^{(i)}-\frac{4}{\sqrt{3}}\phi^{(i)}\right)-2\partial^\nu n_\nu\right)-\partial^\nu\left(\Delta\left(h^{(i)}_{\nu\mu}-\frac{\eta_{\nu\mu}}{\sqrt{3}}\varphi^{(i)}\right)-2\partial_{(\nu}n_{\mu)}\right) && (54)\\
0 &= \Box\varphi^{(i)}+\frac{\sqrt{3}}{2}\Delta\left(\Delta\left(h-\frac{4}{\sqrt{3}}\varphi\right)-2\partial^\alpha n_\alpha\right)_{(i-1)}. && (55)
\end{aligned}
$$

They correspond respectively to linearization of equations (9), (8) and (7).

In order to exhibit the spectrum encoded in the action (51) it is convenient to perform a discrete Fourier transformation. To each field $\mathcal{F}_{(i)}$ with $\mathcal{F}_{(i+N)}=\mathcal{F}_{(i)}$ we define $\hat{\mathcal{F}}_{(k)}$ by

$$
\hat{\mathcal{F}}_{(k)}=\sum_j\frac{1}{\sqrt{N}}\mathcal{F}_{(j)}e^{-i2\pi jk/N}. \tag{56}
$$

The action (51) becomes

$$
\int d^4x\sum_k\frac{1}{4}\left\{\partial_\rho\hat{h}^{\mu\nu}_{(k)}\partial_\sigma\hat{h}^{*\alpha\beta}_{(k)}\left(\eta^{\rho\sigma}\eta_{\mu\nu}\eta_{\alpha\beta}-\eta^{\rho\sigma}\eta_{\mu\alpha}\eta_{\nu\beta}+2\delta^\sigma_{(\nu}\eta_{\mu)\beta}\delta^\rho_\alpha-\eta_{\mu\nu}\delta^\sigma_\beta\delta^\rho_\alpha-\eta_{\alpha\beta}\delta^\sigma_\nu\delta^\rho_\mu\right)\right\}
$$

$$
-\frac{1}{2}\sum_k\partial_\mu\hat{\varphi}^{(k)}\partial_\nu\hat{\varphi}^{*(k)}\eta^{\mu\nu}-\frac{1}{4}(\partial_\mu\hat{n}^{(0)}_\nu-\partial_\nu\hat{n}^{(0)}_\mu)(\partial^\mu\hat{n}^{\nu(0)}-\partial^\nu\hat{n}^\mu_{(0)})+\sum_{k\neq0}\frac{1}{a^2}\sin^2\frac{\pi k}{N}\,\{
$$

$$
\left(\left(\hat{h}^{(k)}_{\mu\nu}-\frac{\eta_{\mu\nu}}{\sqrt{3}}\hat{\varphi}^{(k)}\right)-\frac{2a\partial_{(\mu}\hat{n}^{(k)}_{\nu)}}{e^{i2\pi k/N}-1}\right)\left(\left(\hat{h}^{*(k)}_{\alpha\beta}-\frac{\eta_{\alpha\beta}}{\sqrt{3}}\hat{\varphi}^{*(k)}\right)-\frac{2a\partial_{(\alpha}\hat{n}^{*(k)}_{\beta)}}{e^{-i2\pi k/N}-1}\right)\left(\eta^{\mu\nu}\eta^{\alpha\beta}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)\Bigg\}.
$$

The spin two and one content of the action is easily read from the action. We have one massless spin 2 particle given by $\hat{h}^{(0)}_{\mu\nu}$, one massless spin 1 particle $\hat{n}^{(0)}_\mu$, one massless scalar $\hat{\phi}^{(0)}$ and a tower of massive spin two particles with a spectrum given by

$$
m_k^2=\frac{1}{a^2}\sin^2\frac{\pi k}{N}. \tag{57}
$$

[5] That is to say $\mathcal{F}_{(\mu\nu)}=\frac{1}{2}\left(\mathcal{F}_{\mu\nu}+\mathcal{F}_{\nu\mu}\right)$

The action has the local invariances

$$\delta \hat{h}^{(k)}_{\mu\nu} = 2\partial_{(\mu}\xi^{(k)}_{\nu)}, \quad \delta \hat{n}^{(k)}_{\mu} = \frac{(e^{i2\pi k/N}-1)}{a}\xi^{(k)}_{\mu}, \tag{58}$$

which show that for $k \neq 0$, the $\hat{n}^{(k)}_{\mu}$ are Stuckelberg fields which are absorbed by the massive spin 2 fields and do not propagate. The invariances (58) are the linearized version of the invariance under 4D diffeomorphisms, they are expected by construction. Less expected is the invariance under the local transformations

$$\delta \hat{h}^{(k)}_{\mu\nu} = \eta_{\mu\nu} f^{(k)}, \ \delta\hat{\varphi}^{(k)} = \sqrt{3} f^{(k)}, \ \delta \hat{n}^{(k)}_{\mu} = \frac{a}{1-e^{-i2\pi k/N}}\partial_{\mu} f^{(k)}, \ k \neq 0. \tag{59}$$

A generic multigravity theory with 4D diffeomorphism invariance on each site realized does not possess this symmetry, which is inherited from the diffeomorphism invariance under the y reparametrizations in the continuum theory. In fact the invariance under (59) eliminates, at the quadratic level, all except the massless, scalar modes $\hat{\phi}^{(k)}$. It may also be used to eliminate the trace of the $\hat{h}^{(k)}_{\mu\nu}$ proving the absence of ghostlike excitations. Associated to this local invariance there is a constraint which removes one more set of scalars. At this point we note that while the Pauli-Fierz action removes the ghost by hand, the above action removes it with the aid of a local symmetry; in the gauge where $\hat{\varphi}^{(k)}$ and $\hat{n}^{(k)}_{\mu}$ are zero the action reduces to the Pauli-Fierz form.

The scalar fluctuations are in fact potentially pathologic. This is mainly due to the fact that the conformal factor has a kinetic term with the wrong sign, that is for a metric of the form $g = e^{2\sigma}\eta$, the Einstein action reads

$$6\int (\partial e^{\sigma})^2, \tag{60}$$

giving a ghostlike kinetic term for e^{σ}

In order to explicitly check the consistency of the action (51) it is instructive to isolate in (51) the scalar massive modes which in addition to $\hat{\varphi}^{(k)}$ are given by

$$\hat{\gamma}^{(k)}_{\mu\nu} = \eta_{\mu\nu}\hat{\psi}^{(k)} + \frac{\partial_{\mu}\partial_{\nu}}{\Box}\hat{f}^{(k)}, k \neq 0 \tag{61}$$

$$\hat{n}^{(k)}_{\mu} = \partial_{\mu}\hat{v}^{(k)} \tag{62}$$

At quadratic order in $\hat{\psi}^{(k)}, \hat{f}^{(k)}, \hat{v}^{(k)}$ and $\hat{\phi}^{(k)}$, the action (44) now reads

$$\int d^4x \sum_k \left\{ \frac{3}{2}\left|\partial\hat{\psi}^{(k)}\right|^2 - \frac{1}{2}\left|\partial\hat{\varphi}^{(k)}\right|^2 + \frac{12\sin^2\frac{\pi k}{N}}{a^2}\left|\left(\hat{\psi}^{(k)} - \frac{\hat{\varphi}^{(k)}}{\sqrt{3}}\right)\right|^2 \right.$$
$$\left. + \left[\frac{3}{a^2}\sin^2\frac{\pi k}{N}\left(\hat{f}^{(k)} - \frac{2a\Box\hat{v}^{(k)}}{e^{i2\pi k/N}-1}\right)\left(\hat{\psi}^{(k)} - \frac{\hat{\varphi}^{(k)}}{\sqrt{3}}\right)^* + c.c.\right]\right\} \tag{63}$$

On the above action one clearly sees the ghost like nature of $\hat{\psi}^{(k)}$ as well as the fact that the discretization procedure is leading to tachyonic mass term for $\hat{\phi}^{(k)}$. We saw however that both

problems disappear once one carefully considers the invariances of the action (63) given in (58). The equations of motion derived from the above action (63) now read

$$
\begin{aligned}
\Box\hat{\psi}_{(k)} &= -\frac{8\sin^2\frac{\pi k}{N}}{a^2}\left(\hat{\psi}_{(k)}-\frac{\hat{\varphi}_{(k)}}{\sqrt{3}}\right)-\frac{2\sin^2\frac{\pi k}{N}}{a^2}\left(\hat{f}_{(k)}-\frac{2a\Box\hat{v}^{(k)}}{e^{i2\pi k/N}-1}\right), && (64)\\
\hat{\psi}_{(k)}-\frac{\hat{\varphi}_{(k)}}{\sqrt{3}} &= 0, && (65)\\
\Box\frac{\hat{\varphi_{(k)}}}{\sqrt{3}} &= -\frac{8\sin^2\frac{\pi k}{N}}{a^2}\left(\hat{\psi}_{(k)}-\frac{\hat{\varphi}_{(k)}}{\sqrt{3}}\right)-\frac{2\sin^2\frac{\pi k}{N}}{a^2}\left(\hat{f}_{(k)}-\frac{2a\Box\hat{v}^{(k)}}{e^{i2\pi k/N}-1}\right), && (66)\\
\partial^\mu\left(\hat{\psi}^{(k)}-\frac{\hat{\varphi^{(k)}}}{\sqrt{3}}\right) &= 0. && (67)
\end{aligned}
$$

One can explicitly verify that these equations are obtained from the full equations of motion (53-55). The first three equations are respectively the equation of motion of $\hat{\psi}_{(k)}$, $\hat{f}_{(k)}$, $\hat{\varphi}_{(k)}$, while the last is the equation of motion for $\partial_\mu\hat{v}_{(k)}$ considered as a dynamical variable. Note that had we considered (as in ref. [11]), $\hat{v}_{(k)}$, instead of $\partial_\mu\hat{v}_{(k)}$, as the dynamical variable, we would have obtained the equation of motion $\Box\left(\hat{\psi}_{(k)}-\frac{\hat{\varphi}_{(k)}}{\sqrt{3}}\right)=0$, instead of the constraint (67) which eliminates the modes $\left(\hat{\psi}^{(k)}-\frac{\hat{\varphi}^{(k)}}{\sqrt{3}}\right)$. This would not have been the equation of motion obtained by varying the initial action (5) with respect to N^μ. The constraints (65) when put back in the action leads to the cancellation of the kinetic terms of both $\hat{\psi}^{(k)}$ and $\hat{\varphi}^{(k)}$. It is possible to use the gauge invariance (59) to set $\hat{\varphi}^{(k)}=0$ for $k\neq 0$ and then use the 4D diffeomorphisms to eliminate also $\hat{v}^{(k)}$ and $\hat{f}^{(0)}$. The equations of motion eliminate $\hat{f}^{(k)}$ for $k\neq 0$ as well as $\hat{\psi}^{(k)}$ for all k[6]. The only remaining scalar is thus the massless $\hat{\varphi}^{(0)}$.

At the cubic and higher orders the symmetries (58) are not expected to hold anymore, neither an extension of these. The cubic part of the action is the sum of the cubic part of the Einstein-Hilbert action and the additional terms given by

$$
\begin{aligned}
M_pS^{(3)} = \quad & \sum_i\left(\Delta\left(h^{(i)}_{\mu\nu}-\frac{\eta_{\mu\nu}}{\sqrt{3}}\varphi^{(i)}\right)-2\partial_{(\mu}n^{(i)}_{\nu)}\right)\left(\Delta\left(h^{(i)}_{\alpha\beta}-\frac{\eta_{\alpha\beta}}{\sqrt{3}}\varphi^{(i)}\right)-2\partial_{(\alpha}n^{(i)}_{\beta)}\right)\\
&\left\{\left(\eta^{\mu\nu}\eta^{\alpha\beta}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)\left(\frac{1}{2}h^{(i)}-\frac{\varphi^{(i)}}{\sqrt{3}}\right)-\left(h^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\nu}h^{\alpha\beta}-h^{\mu\alpha}\eta^{\nu\beta}-\eta^{\mu\alpha}h^{\nu\beta}\right)\right\}\\
&+2\left(\Delta\left(h^{(i)}_{\mu\nu}-\frac{\eta_{\mu\nu}}{\sqrt{3}}\varphi^{(i)}\right)-2\partial_{(\mu}n^{(i)}_{\nu)}\right)\left(\eta^{\mu\nu}\eta^{\alpha\beta}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)\\
&\left(-\frac{\Delta(h_{\alpha\beta}\phi)^{(i)}}{\sqrt{3}}+\frac{\eta_{\alpha\beta}}{6}\Delta\phi^2_{(i)}-\frac{\eta_{\alpha\beta}}{\sqrt{3}}n^\lambda_{(i)}\partial_\lambda\phi^{(i+1)}-\frac{2}{\sqrt{3}}\phi^{(i+1)}\partial_{(\alpha}n^{(i)}_{\beta)}\right.\\
&\left.+h_{\alpha_\lambda}\partial_\beta n^\lambda_{(i)}+h_{\beta_\lambda}\partial_\alpha n^\lambda_{(i)}+n^\lambda_{(i)}\partial_\lambda h^{(i+1)}_{\alpha\beta}+a\partial_\alpha n^\lambda_{(i)}\partial_\beta n^{(i)}_\lambda\right). \qquad (68)
\end{aligned}
$$

The characteristic scale of these interactions is given by $\sqrt{NM_p/a}=\sqrt{R}M^{\frac{3}{4}}_{(5)}a^{\frac{-5}{4}}$ when $a>M_p^{-1}$ or else by $\sqrt{N}M_p=M^{\frac{3}{2}}_{(5)}\sqrt{R}$. Notice however, that since at the quadratic order all the scalars

[6] That $\hat{\psi}^{(0)}$ is eliminated can be seen from the full equations (53)

(except $\hat{\varphi}^{(0)}$) decouple and thus have no propagators, we have a non-standard action which starts at a cubic or higher order level for these modes. This deserves a further study.

Acknowledgements

We thank E. Dudas, U. Ellwanger and S. Pokorski for helpful discussions.

References

[1] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A **173** (1939) 211.

[2] D. G. Boulware and S. Deser, Phys. Rev. D **6** (1972) 3368.

[3] C. J. Isham, A. Salam and J. Strathdee, Phys. Rev. D **3** (1971) 867. C. Aragone and J. Chela-Flores, Nuovo Cim. **10A**, 818 (1972); A. Salam and J. Strathdee, Phys. Rev. D **16**, 2668 (1977); C. J. Isham and D. Storey, Phys. Rev. D **18**, 1047 (1978); A. H. Chamseddine, Phys. Lett. B **557**, 247 (2003) [arXiv:hep-th/0301014].

[4] C. Cutler and R. M. Wald, Class. Quant. Grav. **4**, 1267 (1987). R. M. Wald, Class. Quant. Grav. **4**, 1279 (1987).

[5] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, Nucl. Phys. B **597**, 127 (2001) [arXiv:hep-th/0007220].

[6] T. Damour and I. I. Kogan, Phys. Rev. D **66** (2002) 104024 [arXiv:hep-th/0206042]. T. Damour, I. I. Kogan and A. Papazoglou, Phys. Rev. D **66**, 104025 (2002) [arXiv:hep-th/0206044]. T. Damour, I. I. Kogan and A. Papazoglou, Phys. Rev. D **67**, 064009 (2003) [arXiv:hep-th/0212155].

[7] L. Dolan and M. J. Duff, Phys. Rev. Lett. **52** (1984) 14. C. S. Aulakh and D. Sahdev, Phys. Lett. B **164** (1985) 293. C. R. Nappi and L. Witten, Phys. Rev. D **40** (1989) 1095.

[8] M. Reuter, Phys. Lett. B **205**, 511 (1988).

[9] R.Arnowitt, S. Deser, C.W. Misner, "the dynamics of general relativity" in *Gravitation: an introduction to current research*, ed. L. Witten (New York, Wiley, 1962); R. M. Wald, "General Relativity," *Chicago, Usa: Univ. Pr. (1984).*

[10] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, Annals Phys. **305** (2003) 96 [arXiv:hep-th/0210184].

[11] N. Arkani-Hamed and M. D. Schwartz, arXiv:hep-th/0302110; M. D. Schwartz, arXiv:hep-th/0303114; Phys. Rev. D **68**, 024029 (2003)

[12] N. Arkani-Hamed, A. G. Cohen and H. Georgi, Phys. Rev. Lett. **86** (2001) 4757 [arXiv:hep-th/0104005]. C. T. Hill, S. Pokorski and J. Wang, Phys. Rev. D **64** (2001) 105005 [arXiv:hep-th/0104035].

[13] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).

[14] J. B. Kogut, Rev. Mod. Phys. **55**, 775 (1983).

[15] A. Sugamoto, Prog. Theor. Phys. **107**, 793 (2002) [arXiv:hep-th/0104241]; M. Bander, Phys. Rev. D **64**, 105021 (2001) [arXiv:hep-th/0107130]; V. Jejjala, R. G. Leigh and D. Minic, Phys. Lett. B **556**, 71 (2003) [arXiv:hep-th/0212057]. Nahomi Kan and Kiyoshi Shiraishi, Classical and Quantum Gravity 20 (2003) 23, 4965,